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Completeness of coherent states associated with self-similar potentials and Ramanujan's integral extension of the beta function

A N F Aleixo[1], **A B Balantekin**[2] **and M A Cândido Ribeiro**[3]

[1] Instituto de Física, Universidade Federal do Rio de Janeiro, RJ, Brazil
[2] Department of Physics, University of Wisconsin, Madison, WI 53706, USA
[3] Departamento de Física, Instituto de Biociências, Letras e Ciências Exatas, UNESP, São José do Rio Preto, SP, Brazil

E-mail: armando@if.ufrj.br, baha@nucth.physics.wisc.edu and macr@df.ibilce.unesp.br

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Abstract

A decomposition of identity is given as a complex integral over the coherent states associated with a class of shape-invariant self-similar potentials. There is a remarkable connection between these coherent states and Ramanujan's integral extension of the beta function

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1. Introduction

Supersymmetric quantum mechanics is the study of pairs of Hamiltonians with identical energy spectra and with eigenstates that are different, but can be transformed into each other [1, 2]. Some, but not all, such pairs of Hamiltonians share an integrability condition called shape invariance [3]. In [4], it was shown that the shape-invariance condition has an underlying algebraic structure and the associated Lie algebras were identified. Utilizing this algebraic structure a general definition of coherent states for shape-invariant potentials was introduced by several authors [5, 6]. These coherent states are eigenstates of the annihilation operator of the shape-invariant system. When the shape-invariant system is taken to be the standard harmonic oscillator, q-analogue harmonic oscillator or $SU(1, 1)$-covariant system, those coherent states reduce to the well-known harmonic oscillator coherent states, q-analogue coherent states or the Barut–Girardello $SU(1, 1)$ coherent states [7], respectively.

A decomposition of the identity operator using q-integration is available in the literature [8]. In this paper, we use shape-invariant algebraic techniques to obtain a decomposition of the identity as a regular integral over complex variables. It turns out that such an integral is very closely related to an integral evaluated by Ramanujan in studying integral expansion of the beta function.

In the next section, we review the relevant formulae for supersymmetric quantum mechanics and the shape invariance. In section 3, we introduce generalized coherent states for shape-invariant systems. In section 4 we show that, when the appropriate shape-invariant system is chosen, these reduce to the q-analogue coherent states. The proof of completeness of these coherent states in terms of the Ramanujan integral is given in section 5. Several brief remarks in section 6 conclude the paper.

2. Supersymmetric quantum mechanics and shape invariance

In one dimension supersymmetric quantum mechanics uses the operators [1]

$$\hat{A} \equiv W(x) + \frac{\mathrm{i}}{\sqrt{2m}}\hat{p} \tag{2.1}$$

$$\hat{A}^\dagger \equiv W(x) - \frac{\mathrm{i}}{\sqrt{2m}}\hat{p} \tag{2.2}$$

to write the Hamiltonian in the form

$$\hat{H} - E_0 = \hat{A}^\dagger\hat{A}. \tag{2.3}$$

Here, E_0 is the energy of the ground state, the wavefunction of which is annihilated by the operator $\hat{A}$:

$$\hat{A}|\Psi_0\rangle = 0. \tag{2.4}$$

To illustrate the underlying algebraic structure the shape-invariance condition can be written in terms of the operators defined in equations (2.1) and (2.2) as [4]

$$\hat{A}(a_1)\hat{A}^\dagger(a_1) = \hat{A}^\dagger(a_2)\hat{A}(a_2) + R(a_1) \tag{2.5}$$

where $a_{1,2}$ are a set of parameters. The parameter a_2 is a function of a_1 and the remainder $R(a_1)$ is independent of $\hat{x}$ and $\hat{p}$. Introducing the similarity transformation that replaces a_1 with a_2 in a given operator

$$\hat{T}(a_1)\hat{O}(a_1)\hat{T}^\dagger(a_1) = \hat{O}(a_2) \tag{2.6}$$

and the operators

$$\hat{B}_+ = \hat{A}^\dagger(a_1)\hat{T}(a_1) \tag{2.7}$$

$$\hat{B}_- = \hat{B}_+^\dagger = \hat{T}^\dagger(a_1)\hat{A}(a_1) \tag{2.8}$$

the Hamiltonian takes the form

$$\hat{H} - E_0 = \hat{B}_+\hat{B}_-. \tag{2.9}$$

Using equations (2.6)–(2.8) one can show that the commutation relations

$$[\hat{B}_-, \hat{B}_+] = \hat{T}^\dagger(a_1)R(a_1)\hat{T}(a_1) \equiv R(a_0) \tag{2.10}$$

and

$$[\hat{B}_+, R(a_0)] = [R(a_1) - R(a_0)]\hat{B}_+ \tag{2.11}$$

$$[\hat{B}_+, \{R(a_1) - R(a_0)\}\hat{B}_+] = \{[R(a_2) - R(a_1)] - [R(a_1) - R(a_0)]\}\hat{B}_+^2 \tag{2.12}$$

and the Hermitian conjugates of the relations given in equations (2.11) and (2.12) are satisfied [4]. In the most general case the resulting Lie algebra is infinite dimensional.

One class of shape-invariant potentials are reflectionless potentials with an infinite number of bound states, also called self-similar potentials [9, 10]. Shape invariance of such potentials were studied in detail in [11, 12]. For such potentials the parameters are related by a scaling:

$$a_n = q^{n-1} a_1. \tag{2.13}$$

For the simplest case studied in [12] the remainder of equation (2.5) is given by

$$R(a_1) = c a_1 \tag{2.14}$$

where c is a constant and the operator introduced in equation (2.6) by

$$\hat{T}(a_1) = \exp\left\{(\log q) a_1 \frac{\partial}{\partial a_1}\right\}. \tag{2.15}$$

For this shape-invariant potential one can show that the scaled operators

$$\hat{S}_+ = \sqrt{q}\,\hat{B}_+ R(a_1)^{-1/2} \tag{2.16}$$

and

$$\hat{S}_- = (\hat{S}_+)^\dagger = \sqrt{q}\,R(a_1)^{-1/2}\hat{B}_- \tag{2.17}$$

satisfy the standard q-deformed oscillator relation [13]

$$\hat{S}_-\hat{S}_+ - q\hat{S}_+\hat{S}_- = 1. \tag{2.18}$$

The Hamiltonian takes the form

$$\hat{H} - E_0 = R(a_1)\hat{S}_+\hat{S}_-. \tag{2.19}$$

The energy eigenvalues are

$$E_n = R(a_1)\frac{1-q^n}{1-q} \tag{2.20}$$

and the normalized eigenstates are given by

$$|n\rangle = \sqrt{\frac{(1-q)^n}{(q;q)_n}}(\hat{S}_+)^n|0\rangle \tag{2.21}$$

where the q-shifted factorial $(q;q)_n$ is defined as $(z;q)_0 = 1$ and $(z;q)_n = \prod_{j=0}^{n-1}(1-zq^j)$, $n = 1, 2, \ldots$.

One should point out that q-generalizations not only of the standard harmonic oscillator, but also of other exactly solvable problems are available in the literature (see, e.g., [14]). Shape-invariance properties of such generalizations are yet unexplored.

3. Coherent states

Coherent states for shape-invariant potentials were introduced in [5, 6]. Here we follow the notation of [6]. Using the right inverse of $\hat{B}_-$

$$\hat{H}^{-1}\hat{B}_+ = \hat{B}_-^{-1} \qquad (\hat{B}_-\hat{B}_-^{-1} = 1) \tag{3.1}$$

the coherent state for shape-invariant potentials with an infinite number of energy eigenstates was defined in [6] as

$$\begin{aligned} |z\rangle_c &= |0\rangle + z\hat{B}_-^{-1}|0\rangle + z^2\hat{B}_-^{-2}|0\rangle + \cdots \\ &= \frac{1}{1-z\hat{B}_-^{-1}}|0\rangle \end{aligned} \tag{3.2}$$

where $|0\rangle$ is the ground state of the Hamiltonian in equation (2.9). The coherent state in equation (3.2) is an eigenstate of the operator $\hat{B}_-$:

$$\hat{B}_-|z\rangle_c = z|z\rangle_c. \tag{3.3}$$

In this paper, we use a slightly more general definition of the coherent state. Introducing an arbitrary functional $f[R(a_1)]$ of the remainder in equation (2.5), we define the coherent state to be

$$|z\rangle = \sum_{n=0}^{K} \left(zf[R(a_1)]\hat{B}_-^{-1}\right)^n |0\rangle \tag{3.4}$$

where the upper limit K in the sum, depending on the nature (the number of eigenstates) of the potential, can be either finite or infinite. Note that one can define a new variable $z' = zf(a_1)$, i.e. this generalization permits z to be a function of the variables $a_1, a_2, a_3, \ldots$. One can write equation (3.4) explicitly as

$$|z\rangle = |0\rangle + z\,\frac{f[R(a_1)]}{\sqrt{R(a_1)}}|1\rangle + z^2\frac{f[R(a_1)]f[R(a_2)]}{\sqrt{R(a_2)[R(a_1)+R(a_2)]}}|2\rangle$$
$$+ z^3\frac{f[R(a_1)]f[R(a_2)]f[R(a_3)]}{\sqrt{R(a_3)[R(a_2)+R(a_3)][R(a_3)+R(a_2)+R(a_1)]}}|3\rangle + \cdots \tag{3.5}$$

where $|n\rangle$ is the nth excited state of the system:

$$|n\rangle = [\hat{H}^{-1/2}\hat{B}_+]^n|0\rangle. \tag{3.6}$$

(cf with equation (2.21)).

If the ground state is normalized, i.e. $\langle 0 \mid 0\rangle = 1$, then all the excited states given by equation (3.6) are normalized as well. If the number of energy eigenstates is infinite the coherent state defined in equation (3.4) is also an eigenstate of the operator $\hat{B}_-$:

$$\hat{B}_-|z\rangle = zf[R(a_0)]|z\rangle. \tag{3.7}$$

The additional condition

$$[\hat{B}_- - zf[R(a_0)]]\frac{\partial}{\partial z}|z\rangle = f[R(a_0)]|z\rangle \tag{3.8}$$

is also satisfied.

4. *q*-coherent states for shape-invariant systems

We now show that when $R(a_1)$ is given by equation (2.14) and is subject to the scaling transformation given in equation (2.13) the coherent state of equation (3.4) reduces to the standard q-coherent states introduced in [15, 16]. In this case, equation (3.4) can be written in the form

$$|z\rangle = \sum_{n=0}^{\infty}(f[R(a_1)]f[R(a_2)]\cdots f[R(a_n)])\frac{(1-q)^{n/2}}{\sqrt{(q;q)_n}}q^{-n(n-1)/4}\frac{z^n}{\sqrt{[R(a_1)]^n}}|n\rangle. \tag{4.1}$$

We now choose $f[R(a_n)] = R(a_n)$. The coherent state of equation (4.1) takes the form

$$|z\rangle = \sum_{n=0}^{\infty}\frac{(1-q)^{n/2}}{\sqrt{(q;q)_n}}q^{n(n-1)/4}\sqrt{[R(a_1)]^n}z^n|n\rangle. \tag{4.2}$$

Using the normalized eigenstates given in equation (2.21), the above equation can be written as

$$|z\rangle = \sum_{n=0}^{\infty} \frac{(1-q)^n}{(q;q)_n} q^{n(n-1)/4} \left(z\sqrt{[R(a_1)]}\hat{S}_+\right)^n |0\rangle. \tag{4.3}$$

Using the q-exponential utilized in [17] (see also [18, 19])

$$E_q^{(\mu)}(x) = \sum_{n=0}^{\infty} \frac{q^{\mu n^2}}{(q;q)_n} x^n \tag{4.4}$$

and the identity $\sqrt{[R(a_1)]}\hat{S}_+ = \hat{B}_+$, the coherent state can be written as a generalized exponential:

$$|z\rangle = E_q^{(1/4)}\left(\frac{(1-q)}{q^{1/4}} z\hat{B}+\right)|0\rangle. \tag{4.5}$$

Finally, introducing the variable

$$\zeta = \frac{\sqrt{(1-q)}}{\sqrt{q}}\sqrt{R(a_1)}z \tag{4.6}$$

the coherent state can also be written as

$$|\zeta\rangle = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/4}}{\sqrt{(q;q)_n}} \zeta^n |n\rangle. \tag{4.7}$$

The norm of this state is also given by a generalized exponential:

$$\langle\zeta|\zeta\rangle = E_q^{(1/2)}(q^{1/4}|\zeta|^2). \tag{4.8}$$

Equation (4.8) is useful when writing path integrals in coherent-state representation.

5. Completeness of *q*-coherent states and Ramanujan integrals

In this section, we investigate the completeness relation for q-analogue coherent states given in the previous section. One way to obtain a decomposition of identity using q-analogue coherent states is to use Jackson's formula for the q-differentiation and q-integration [20]. Such a completeness relation was introduced by the authors of [8] (see also [21]). In the current work we introduce a completeness relation *without* using q-integration.

Our main result is that the decomposition of identity for the coherent states of equation (4.7) is given by

$$I = \int \frac{\mathrm{d}\zeta\,\mathrm{d}\zeta^*}{2\pi\mathrm{i}} \frac{1}{(-\log q)} \frac{1}{(-|\zeta|^2;q)_\infty} |\zeta\rangle\langle\zeta| = \hat{1} \tag{5.1}$$

which we prove in this section. (In the above relation $\hat{1}$ is the identity operator in the Hilbert space of the q-analogue harmonic oscillator.)

We change the variables $\zeta = \sqrt{t}\,\mathrm{e}^{\mathrm{i}\theta}$, using equation (4.7) to write the coherent states in terms of the eigenstates of the Hamiltonian, and perform the θ-integration to obtain

$$I = \frac{1}{(-\log q)} \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n} |n\rangle\langle n| \times \int_0^\infty \mathrm{d}t \frac{t^n}{(-t;q)_\infty}. \tag{5.2}$$

The last integral was evaluated by Ramanujan in an attempt to generalize integral definition of the beta function [22]. (An elementary proof is given by Askey [23].) It is given by

$$\int_0^\infty \mathrm{d}t \frac{t^n}{(-t;q)_\infty} = \frac{(q;q)_n}{q^{n(n+1)/2}}(-\log q). \tag{5.3}$$

Inserting equation (5.3) into equation (5.2) provides the proof of equation (5.1):

$$I = \sum_{n=0}^{\infty} |n\rangle\langle n| = \hat{1}. \tag{5.4}$$

As we mentioned above another decomposition of identity for q-coherent states is given in [8]. In this reference a q-analogue of the Euler formula for $\Gamma(z)$ expressed as a q-integral is utilized instead of the Ramanujan integral, equation (5.3). Consequently, the authors of [8] obtained the resolution of identity for q-analogue harmonic oscillators as a q-integral in contrast to our result, equation (5.4), where this resolution is expressed as an ordinary integral over complex variables.

6. Conclusions

In this paper we presented an overcompleteness relation for the q-analogue coherent states as a complex integral. In arriving at this result we utilized shape-invariant algebraic techniques. Such a result could be very useful in building coherent states of path integrals [24] for q-analogue systems. Of course, such path integrals can also be defined over q-integrals. However, if one wishes to approximate these path integrals using saddle-point approximations or numerically evaluate them using Monte Carlo techniques, ordinary path integrals defined over complex variables present several advantages over the q-integrals. Among these are the presence of a continuous path to find stationary phases and the existence of positive-definite probabilities to do Monte Carlo integration.

Coherent states for the ordinary harmonic oscillator have a dynamical interpretation. If a forced harmonic oscillator is in its ground state for $t = 0$, it evolves into the harmonic oscillator coherent state. As was shown in [6], the coherent states described here, in general, do not have such a simple dynamical interpretation. However, the evolution operator for the q-analogue of the forced harmonic oscillator Hamiltonian

$$\hat{h}(t) = \hat{B}_+\hat{B}_- + f(t)[\mathrm{e}^{\mathrm{i}R(a_1)t/\hbar}\hat{B}_+ + \hat{B}_-\,\mathrm{e}^{-\mathrm{i}R(a_1)t/\hbar}] \tag{6.1}$$

where $f(t)$ is an arbitrary function of time, can be written as a path integral where the integration path is given by the time dependence of the variable ζ of equation (5.1) (cf [24] for the forced harmonic oscillator).

Our decomposition of identity can easily be generalized to multi-dimensional q-oscillators and can be used to study, for example, the dynamics of $SL_q(n)$, which can be constructed from such oscillators using a q-deformed Levi-Civita tensor [25]. (For a review of various applications of quantum groups see, e.g., [26].) Coherent states for harmonic oscillator representation of orthosymplectic superalgebras along with their overcompleteness relations are given in [27, 28]. (For an application of such representations see [29].) Our techniques can be utilized to determine the integration measures for the q-extensions of superalgebras.

Askey reports that Hardy mentions the Ramanujan integral of equation (5.3) to be new and interesting [23]. Askey further remarks that this integral is even more interesting than Hardy seems to have thought it was and points out that it does not seem to have arisen in applications very often. As we demonstrated, this integral seems to play a major role in building the Hilbert space for the q-oscillator Hamiltonian.

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